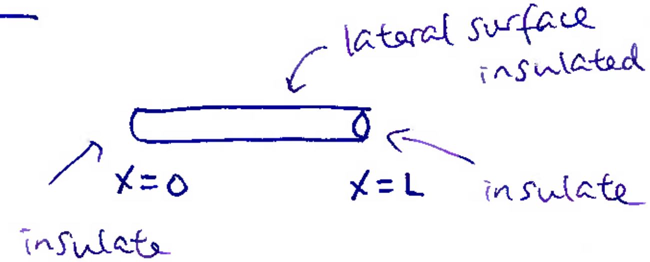
Heat Eq. (part 3)

$u_t = k u_{xx} \qquad 0 < x < L \qquad t > 0$



now we will insulate the ends

heat cannot flow through the ends

this changes the boundary conditions : $\left.\begin{array}{l} u_x(0,t) = 0 \\ u_x(L,t) = 0 \end{array}\right\}$ Neumann BC's

same initial condition as before : $u(x,0) = f(x)$

same method : separation of variables

$$u(x,t) = X(x)\,T(t)$$

$$u_t = X T' \qquad\qquad u_{xx} = X'' T$$

$$u_t = k u_{xx} \longrightarrow X T' = k X'' T$$

$$\frac{X''}{X} = \frac{T'}{KT} = -\lambda \quad \text{separation constant}$$

two ODE's result: $X'' + \lambda X = 0$

$T' + K\lambda T = 0$

BC's: $u_x(0,t) = 0 \rightarrow X'(0)T(t) = 0 \rightarrow X'(0) = 0$

$u_x(L,t) = 0 \rightarrow X'(L)T(t) = 0 \rightarrow X'(L) = 0$

let's solve $X'' + \lambda X = 0 \qquad X'(0) = X'(L) = 0$

$$X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \qquad (\lambda \neq 0)$$

$$X'(x) = -\sqrt{\lambda}A\sin(\sqrt{\lambda}x) + \sqrt{\lambda}B\cos(\sqrt{\lambda}x)$$

$$X'(0) = 0 = \sqrt{\lambda}B \rightarrow B = 0$$

$$X'(L) = 0 = -\sqrt{\lambda}A\sin(\sqrt{\lambda}L)$$

require $A \neq 0$

$\sin(\sqrt{\lambda} L) = 0$ $\qquad \sqrt{\lambda} L = n\pi$ $\qquad \boxed{n = 1, 2, 3, \ldots}$

for each n, $\boxed{\lambda_n = \dfrac{n^2\pi^2}{L^2}}$ eigenvalue

for each n, there is one solution

$$\boxed{X_n = \cos\left(\frac{n\pi}{L} x\right)}$$

eigenfunctions

(cosine instead of sine)

(drop the scaling constant A)

what if $\lambda = 0$ in $X'' + \lambda X = 0$ $\qquad X'(0) = X'(L) = 0$

$$X = Ax + B$$

$$X' = A$$

$$X'(0) = X'(L) = 0 \rightarrow A = 0$$

so, the solution is multiple of 1 (drop B the scaling constant)

$\boxed{X = 1}$ if $\boxed{\lambda = 0}$

notice these are included in the previous case for $n = 0$

therefore solutions are $\boxed{X_n = \cos\left(\frac{n\pi}{L}x\right)}$ $\boxed{n = 0, 1, 2, 3, \ldots}$

$$\boxed{\lambda_n = \frac{n^2\pi^2}{L^2}}$$

now $T' + k\lambda T = 0$ (exactly the same as in the non-insulated case)

$$\boxed{T_n = e^{-kn^2\pi^2 t/L^2}} \qquad n = 0, 1, 2, 3, \ldots$$

general solution is linear combination of all

$$u_n = X_n T_n$$

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$$

$$= A_0 + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$$

at $t = 0$, $u(x,0) = f(x)$

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$ cosine series (missing $1/2$)

we will artificially include a factor of $\frac{1}{2}$ for $n=0$

$$\boxed{u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)}$$

$u(x,0) = f(x)$

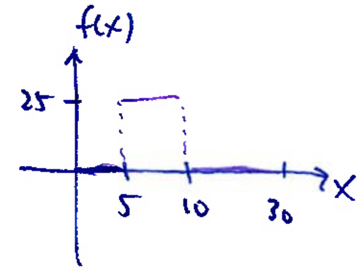
$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

standard cosine series

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

<u>example</u> $L=30$ $\quad k=1$ $\quad$ insulated ends

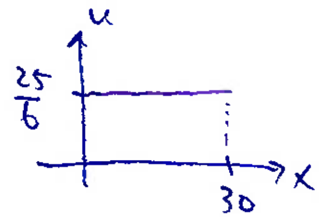
initial condition: $f(x) = \begin{cases} 0 & 0<x<5 \\ 25 & 5<x<10 \\ 0 & 10<x<20 \end{cases}$



$\vdots$

$$u(x,t) = \frac{25}{6} + \sum_{n=1}^{\infty} \frac{50}{n\pi}\left[\sin\left(\frac{n\pi}{3}\right) - \sin\left(\frac{n\pi}{6}\right)\right] e^{-n^2\pi^2 t/900} \cos\left(\frac{n\pi x}{30}\right)$$

Steady-state ($t \to \infty$) $u = \frac{25}{6}$

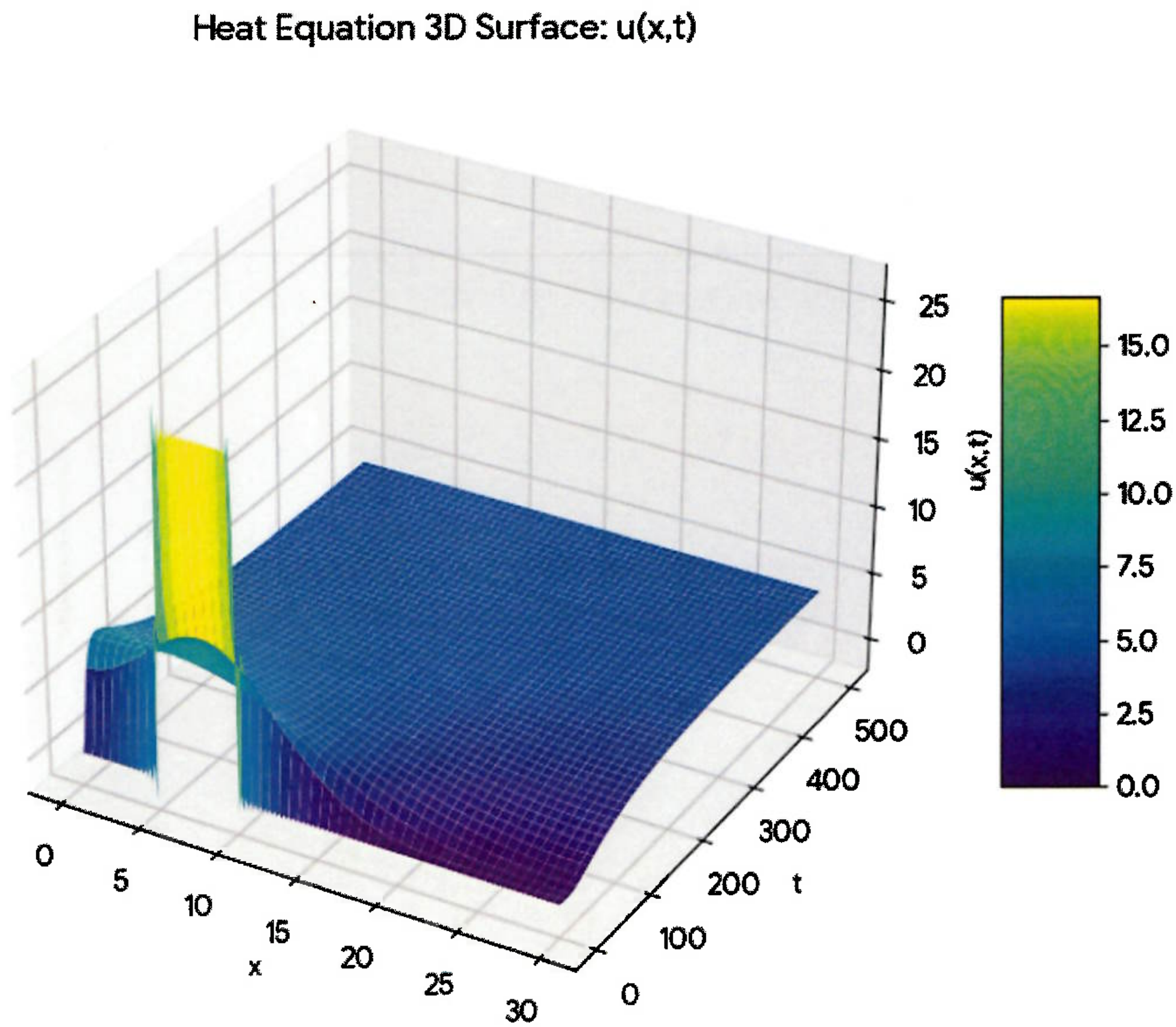


this is the average initial value (area under the first graph is the same as area under the second)

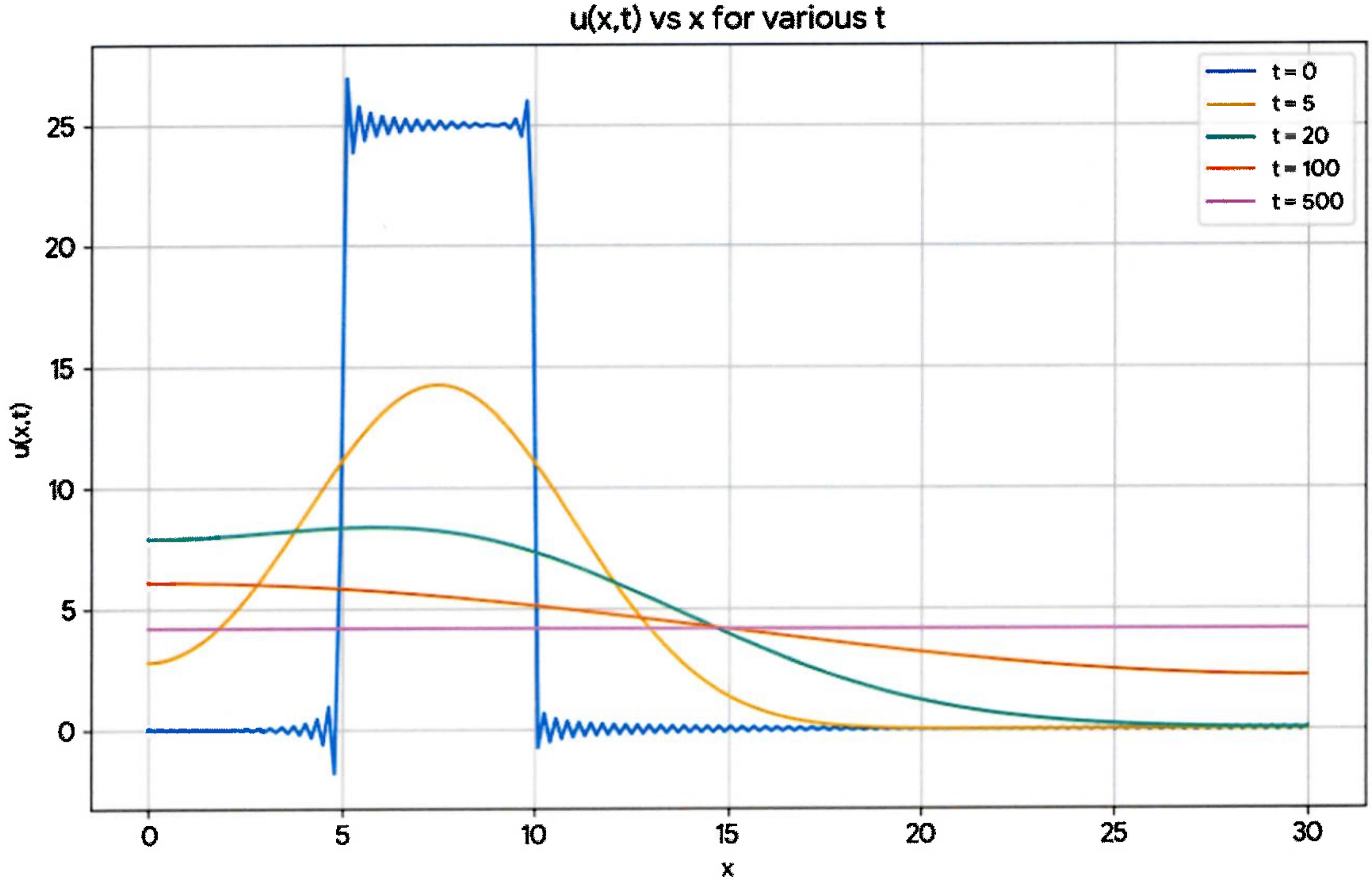
→ heat cannot leave or enter after $t = 0$

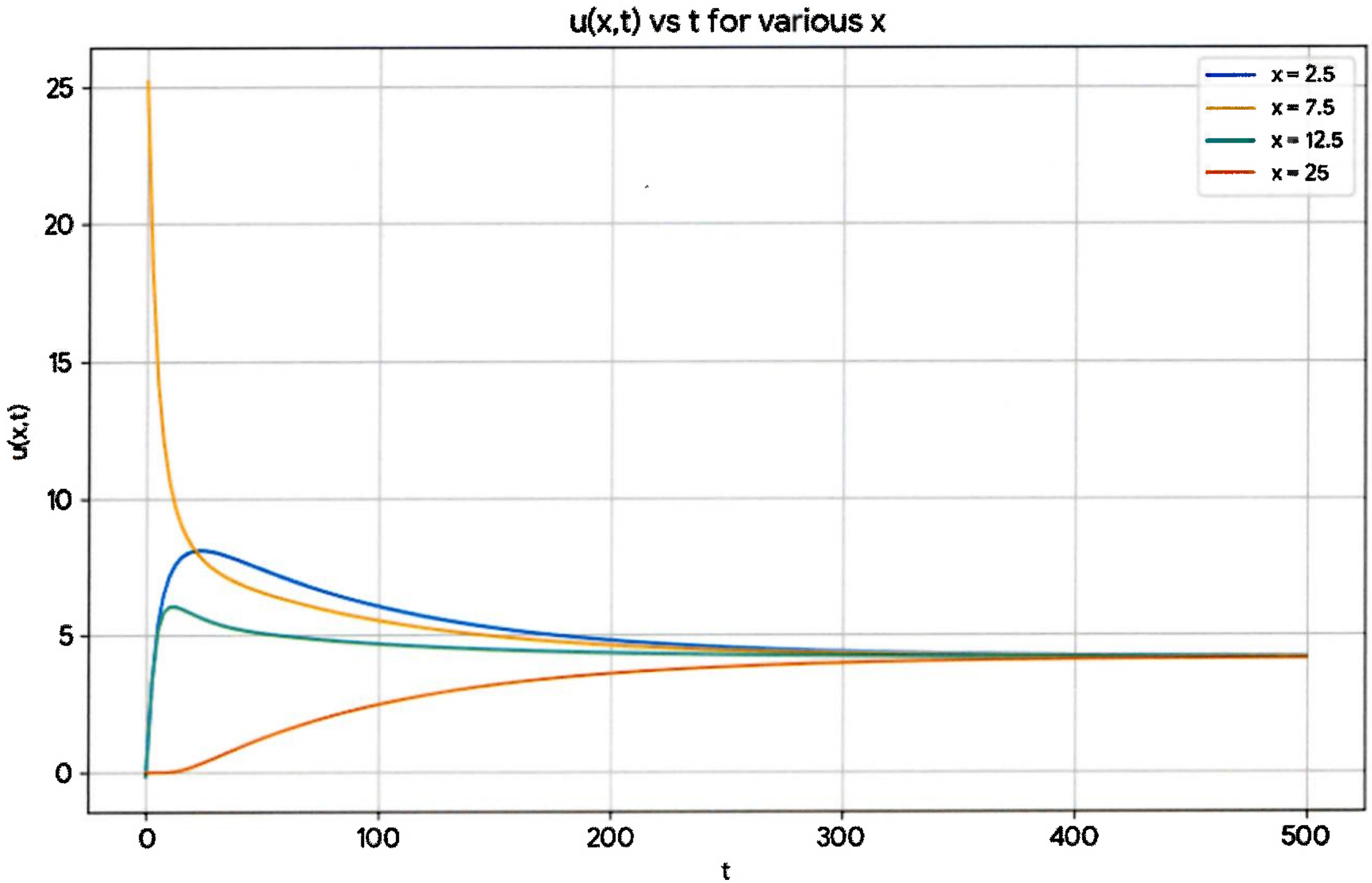
→ heat doesn't want to be uneven (no one spot hotter/colder than nearby average)

→ heat also doesn't want any concavity ($u_{xx} = 0$ as $t \to \infty$)



Heat Equation 3D Surface: u(x,t)
x
0
5
10
15
20
25
30
t
0
100
200
300
400
500
0
5
10
15
20
25
u(x,t)
0.0
2.5
5.0
7.5
10.0
12.5
15.0


u(x,t) vs x for various t
u(x,t)
x
t = 0
t = 5
t = 20
t = 100
t = 500
0
5
10
15
20
25
30


u(x,t) vs t for various x
x = 2.5
x = 7.5
x = 12.5
x = 25
25
20
15
10
5
0
u(x,t)
0
100
200
300
400
500
t